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HAUSDORFF DIMENSION AND DISTANCE SETS

BY

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ABSTRACT

According to a result of K. Falconer (1985), the set $D(A) = \{|x-y|; x, y \in A\}$ of distances for a Souslin set A of $\mathbb{R}^n$ has positive 1-dimensional measure provided the Hausdorff dimension of A is larger than $(n+1)/2$.* We give an improvement of this statement in dimensions $n = 2, n = 3$. The method is based on the fine theory of Fourier restriction phenomena to spheres. Variants of it permit further improvements which we don't plan to describe here. This research originated from some discussions with P. Mattila on the subject.

Here is a sketch of the following.

Let $A \subset \mathbb{R}^2$ have $\dim A > \frac{13}{9}$. Then $\{|x-y|; x, y \in A\}$ has positive measure

(a) Take $\rho < \dim A$. There is a probability measure μ on A satisfying

$$\iint \frac{\mu(dx)\mu(dy)}{|x-y|^\rho} < \infty \tag{1}$$

and hence

$$\int_{|\xi|>1} \frac{|\hat{\mu}(\xi)|^2}{|\xi|^{2-\rho}} d\xi < \infty. \tag{2}$$

Let B be a ball of radius M in $\mathbb{R}^2$. Then (2) implies

$$\int_B |\hat{\mu}(\xi)|^2 d\xi \lesssim M^{2-\rho}. \tag{3}$$

* $\dim A > n/2$ would be the optimal result for $n \geq 2$.

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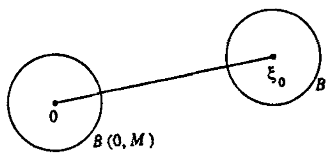
This is clear if B is within $B(0,10M)$. Otherwise, consider a function φ satisfying

$$\begin{aligned}
&\varphi \geq 0, \quad \hat{\varphi} \geq 0, \\
&\hat{\varphi} \sim 1 \quad \text{on } B(0,M), \\
&\hat{\varphi} = 0 \quad \text{outside } (0,2M).
\end{aligned}$$



Since $f(x) = (1-\cos\langle x,\xi_0\rangle)\varphi(x)$ is positive, one gets from the conditions on φ

$$\begin{aligned}
0 &\leq \int f\,d(\mu*\mu) = \int \hat{f}(\xi)|\hat{\mu}(\xi)|^2 d\xi \\
&= \int \hat{\varphi}(\xi)|\hat{\mu}(\xi)|^2 - \frac{1}{2}\int [\hat{\varphi}(\xi+\xi_0)+\hat{\varphi}(\xi-\xi_0)]|\hat{\mu}(\xi)|^2 \\
&\leq c\int_{B(0,2M)} |\hat{\mu}(\xi)|^2 - c\int_B |\hat{\mu}(\xi)|^2.
\end{aligned}$$

Hence

$$\int_B |\hat{\mu}(\xi)|^2 \leq c\int_{B(0,2M)} |\hat{\mu}(\xi)|^2 \leq M^{2-\rho}.$$

(b) Let σ_s be the arc length measure of $S^1(s)$. It suffices to establish an inequality $(0<\alpha<\beta<\infty)$

$$\int_\alpha^\beta |\langle \mu, \mu*\sigma_s\rangle|^2 ds < \infty \tag{4}$$

expressing that the image measure $\delta(\mu)$ of $\mu\times\mu$ under the distance map $|x-y|$ has an L^2-density.

Writing

$$\langle \mu,\mu*\sigma_s\rangle \sim \int |\hat{\mu}(\xi)|^2 \frac{e^{is|\xi|}}{|\xi|^{1/2}} d\xi = \iint |\hat{\mu}(re^{i\theta})|^2 r^{1/2} e^{isr}\,dr\,d\theta,$$

(4) amounts to (from Parseval)

$$\int \left[\int |\hat{\mu}(re^{i\theta})|^2 d\theta\right]^2 r\,dr < \infty. \tag{5}$$

Our next aim is to bound the quantities (cf. [M])

$$\sigma(\mu)(r) \equiv \int |\hat{\mu}(re^{i\theta})|^2 d\theta \tag{6}$$

as a function of r for $r \to \infty$.

Assume we have proved that (6) $\leq r^{-\lambda}$. Then (5) $\leq \int |\xi|^{-\lambda} |\hat{\mu}(\xi)|^2 d\xi$ and one may conclude (5) from (2), provided

$$\lambda > 2 - \rho. \tag{7}$$

(c) By taking a suitable convolution of μ, we get a function $F \geq 0$ depending on r and satisfying by (2)

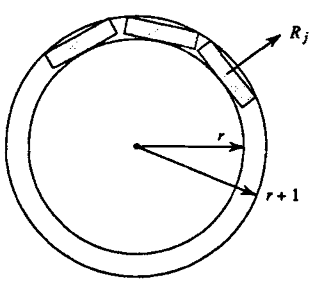
$$|\hat{F}(\xi)| \leq |\hat{\mu}(\xi)|, \quad \int_{\mathbb{R}^2} |F|^2 \leq r^{2-\rho} \tag{8}$$

and

$$(6) \sim \int \left|\hat{F}(re^{i\theta})\right|^2 d\theta \sim \int_r^{r+1} \int \left|\hat{F}(r_1 e^{i\theta})\right|^2 dr_1\, d\theta. \tag{9}$$

We will use some techniques from the restriction theory of Fourier transforms to spheres and the Bochner–Riesz problem (the geometric approach).

Consider the annulus A



which we roughly reconstruct as union of $\sim \sqrt{r}$ rectangles R_j of measurements $1 \times \sqrt{r}$. Thus (9) becomes

$$\frac{1}{r}\left\langle \hat{F}, \hat{F}\chi_A \right\rangle = \frac{1}{r}\langle F, F * \hat{\chi}_A\rangle \leq \frac{1}{r}\|F\|_{L^{4/3}(\mathbb{R}^2)} \left\| \sum_j \left(\hat{F}\cdot \chi_{R_j}\right)^{\vee} \right\|_{L^4(\mathbb{R}^2)} \tag{10}$$

One has $\|F\|_1 \leq 1$ and $\|F\|_2 \leq r^{1-\rho/2}$ by (8). Thus interpolation yields

(11) $$\|F\|_{4/3} \leq r^{\frac{1}{2}-\frac{\rho}{4}}.$$

For the second factor of (10), one follows the Cordoba–Fefferman argument, reducing the problem to the Kakeya maximal function in the plane. Thus

(12)
$$\left\|\sum_j \left(\hat{F}\chi_{R_j}\right)^{\vee}\right\|_4 \sim \left\|\left(\sum_j \left|\left(\hat{F}\cdot\chi_{R_j}\right)^{\vee}\right|^2\right)^{1/2}\right\|_4 \text{ (square function equivalence)}$$
$$= \left\|\sum_j \left|\left(\hat{F}\chi_{R_j}\right)^{\vee}\right|^2\right\|_2^{1/2}$$

Define

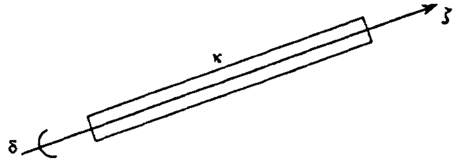
(13) $$c_j = \int \left|\left(\hat{F}\chi_{R_j}\right)^{\vee}\right|^2 = \int_{R_j} |\hat{F}(\xi)|^2,$$

(14) $$\tau_j = \frac{\left|\left(\hat{F}\chi_{R_j}\right)^{\vee}\right|^2}{c_j}.$$

Roughly speaking, the function τ_j may be obtained as a convex combination of translates of the function $\chi_{R_j^0}/|R_j^0|$ where R_j^0 stands for the geometric "polar" of R_j, thus a $(1 \times \frac{1}{\sqrt{r}})$-rectangle perpendicular on R_j and $|R_j^0|$ denotes its measure (see [B1], section 6, for more details on these matters). Estimate $\left\|\sum_j c_j\tau_j\right\|_2$ by duality. Consider the maximal function

(15) $$\mathcal{M}_\delta f(\zeta) = \sup \frac{1}{|\kappa|}\int_\kappa |f|$$

defined for $\zeta \in S^1$, where the sup is taken over all rectangles κ directed along ζ with unit length and with δ



One has the inequality (for this "Kakeya" maximal operator; see, for instance,

[F2], Section 7)

$$\|\mathcal{M}_\delta f\|_{L^2(S^1)} \le \left(\log\frac{1}{\delta}\right)^{1/2} \|f\|_{L^2(\mathbb{R})}. \tag{16}$$

One gets from the preceding for $\|f\|_2 \le 1$

$$\begin{aligned}\left\langle \sum_j c_j\tau_j, f\right\rangle &\le \sum_j c_j(\mathcal{M}_{1/\sqrt{r}}f)(\zeta_j)\\ &\le \left(\sum_j c_j^2\right)^{1/2} r^{1/4}\left(\sum_j \frac{1}{\sqrt{r}}|(\mathcal{M}_{1/\sqrt{r}}f)(\zeta_j)|^2\right)^{1/2}\end{aligned} \tag{17}$$

where $\{\zeta_j | j = 1, \dots, \sqrt{r}\}$ is a $\sim \frac{1}{\sqrt{r}}$ separated set on S^1. Hence

$$\begin{aligned}(17) &\sim \left(\sum_j c_j^2\right)^{1/2} r^{1/4}\|\mathcal{M}_{1/\sqrt{r}}f\|_2\\ &\le (\log r)^{1/2}\cdot r^{1/4}\cdot \max c_j^{1/2}\cdot\left(\sum_j c_j\right)^{1/2}.\end{aligned}$$

Since R_j is contained in a ball of radius $\sqrt{r}$, (3) and (8) imply

$$c_j \le r^{1-\frac{\rho}{2}}. \tag{18}$$

Collecting estimates (9), (10), (11), (17), (12), (18), it follows that

$$(9) \le r^{-1}r^{\frac{1}{2}-\frac{\rho}{4}}\cdot(\log r)^{\frac{1}{4}}r^{1/8}r^{\frac{1}{4}-\frac{\rho}{8}}\left(\sum_j c_j\right)^{1/4} \tag{19}$$

where in fact

$$\sum_j c_j = \int_{\cup R_j}|\hat{F}(\xi)|^2 \le \int_A |\hat{F}(\xi)|^2 = (9)\cdot r.$$

Thus from (19)

$$(9)^{3/4} \ll r^{\frac{1}{8}-\frac{3\rho}{8}+\varepsilon} \Rightarrow (9) \ll r^{\frac{1}{6}-\frac{\rho}{2}+\varepsilon}. \tag{20}$$

Thus in estimating (6), one may take any

$$\lambda < \frac{\rho}{2} - \frac{1}{6} \tag{21}$$

and condition (7) becomes $\rho > \frac{13}{9}$, as claimed.

(d) In the higher-dimensional case, there is an improvement on $\frac{n+1}{2}$ using previous reasoning and the results from my previously mentioned paper, at least for $n = 3$.

(2) becomes

$$\int_{|\xi|>1} \frac{|\hat{\mu}(\xi)|^2}{|\xi|^{n-\rho}} d\xi < \infty \tag{22}$$

and (5)

$$\int \left[\int_{S_{n-1}} |\hat{\mu}(r\zeta)|^2 d\zeta \right]^2 r^{n-1} dr < \infty. \tag{23}$$

In order to deduce (23) from (22), one needs the estimate

$$\sigma(\mu)(r) \equiv \int_{S_{n-1}} |\hat{\mu}(r\zeta)|^2 d\zeta \ll r^{\rho-n}. \tag{24}$$

Let us just use L^2-restriction theory to bound the left member of (24). The L^2 restriction exponent is $\frac{2(n+1)}{n+3} = p$.* The function F satisfies $\|F\|_{L^2(\mathbb{R}^n)} \leq r^{\frac{n-\rho}{2}}$ and hence $\|F\|_p \leq r^{\frac{n-1}{n+1}\frac{n-\rho}{2}}$. Define $F_r(x) = \frac{1}{r^n} F(\frac{x}{r})$, so that $\hat{F}_r(\xi) = \hat{F}(r\xi)$,

$$(24) \sim \left\| \hat{F}_r|_S \right\|^2_{L^2(S_{n-1})} \leq c\|F_r\|_p^2, \tag{25}$$

$$\|F_r\|_p = r^{-\frac{n}{p'}} \|F\|_p \leq r^{-\rho\frac{n-1}{2(n+1)}}. \tag{26}$$

From (25), (26) condition (24) yields

$$\begin{aligned} \rho\frac{n-1}{n+1} &> n - \rho, \\ \rho &> \frac{n+1}{2}. \end{aligned} \tag{27}$$

(e) Using the (partial) knowledge of the L^q-restriction phenomena for $q < 2$ from [B1], [B2], a small improvement on (27) may be derived, in dimension $n = 3$. We proceed as follows. Write

$$\left\| \hat{F}_r|_S \right\|^2_{L^2(S_2)} = \left\langle F_r, F_r * \check{\sigma}_2 \right\rangle \leq \|F_r\|_{4/3} \left\| F_r * \check{\sigma}_2 \right\|_4. \tag{28}$$

* See [St] for a survey on this subject.

Next recall an inequality from [B2] (see lemma 3.23) which in the present context implies for

$$\nu = \hat{F}_r \cdot \sigma_2, \tag{29}$$

$$\begin{aligned} \|\hat{\nu}\|_{L^4(\mathbb{R}^3)} &\le \log r \left(\sum_{\substack{\delta \text{ dyadic } <1 \\ \tau \in C_\delta}} \delta^4 \left\| \frac{d\nu}{d\sigma} \right\|^4_{L^{16/9}(\tau)} \right)^{1/4} \\ &\le (\log r)^2 \sup_{\frac{1}{r} < \delta < 1} \left(\sum_{\tau \in C_\delta} \delta^4 \left\| \frac{d\nu}{d\sigma} \right\|^4_{L^{16/9}(\tau)} \right)^{1/4}. \end{aligned} \tag{30}$$

We give some explanations related to (30). For each $0 < \delta < 1, C_\delta$ denotes a collection of δ-caps of S_2 forming a covering of bounded multiplicity. $L^s(\tau)$ refers to the L^s-norm on τ equipped with normalized measure.

Another inequality (see [B1], [B2], Prop. 2.15) expresses a restriction-extension phenomenon beyond the L^2-theory.

For $4 > \bar{p} > \frac{58}{15}$, there is the inequality (see [B1])

$$\|\hat{\nu}\|_{L^{\bar{p}}(\mathbb{R}^3)} \le c \left\| \frac{d\nu}{d\sigma} \right\|_{L^{\bar{p}}(S_2)}. \tag{31}$$

Recall also from inequality (22) the bound

(32)
$$\int_\tau \left|\hat{F}_r(\xi)\right|^2 \sigma(d\xi) = \int_\tau \left|\hat{F}(r\xi)\right|^2 \sigma(d\xi) = \frac{1}{r^2} \int_{r\cdot\tau + B(0,1)} \left|\hat{F}(\zeta)\right|^2 d\zeta \le \frac{1}{r^2} (r\cdot\delta)^{3-\rho}$$

where $\tau \subset S_2$ is a δ-cap, $\delta > 1/r$. The last inequality in (32) follows from the fact that $\int_B |\hat{\mu}(\xi)|^2 d\xi \le M^{3-\rho}$ for B an M-ball in $\mathbb{R}^3$. The proof is analogous to (3).

It follows from (30) that

(33)
$$\begin{aligned} \left\| F_r * \overset{\vee}{\sigma}_2 \right\|_4 &\le (\log r)^2 \left(\max_{\substack{1/r<\delta<1 \\ \tau\in C_\delta}} \delta^{1/2} \left\| \frac{d\nu}{d\sigma} \right\|^{1/2}_{L^{16/9}(\tau)} \right) \sup_\delta \left(\sum_{\tau \in C_\delta} \delta^2 \left\| \frac{d\nu}{d\sigma} \right\|^2_{L^2(\tau)} \right)^{1/4} \\ &\le (\log r)^2 \left\| \frac{d\nu}{d\sigma} \right\|^{1/2}_{L^2(S_2)} \left(\max_{\substack{\frac{1}{r}<\delta<1 \\ \tau\in C_\delta}} \delta \left\| \frac{d\nu}{d\sigma} \right\|_{L^{16/9}(\tau)} \right)^{1/2} \end{aligned}$$

where $\frac{d\nu}{d\sigma} = \hat{F}_r|_S$.

Hence from (28), (26)

$$\left\|\hat{F}_r|_S\right\|_{L^2(S)} \le (\log r)^2 \|F_r\|_{4/3}^{2/3} \left(\max \delta \left\|\frac{d\nu}{d\sigma}\right\|_{L^{16/9}(\tau)}\right)^{1/3} \tag{34}$$

$$\le (\log r)^2 r^{-\rho/6} \left(\max\ \delta \cdot \left\|\frac{d\nu}{d\sigma}\right\|_{L^{16/9}(\tau)}\right)^{1/3}. \tag{35}$$

We will next estimate the last factor of (35). First, from (32)

$$\left(\int_\tau \left|\hat{F}_r(\xi)\right|^2 \sigma(d\xi)\right)^{1/2} \le r^{\frac{1-\rho}{2}} \delta^{\frac{3-\rho}{2}}. \tag{36}$$

Next, we exploit (31). Writing

$$\frac{1}{\frac{16}{9}} = \frac{1-\theta}{2} + \frac{\theta}{\bar{p}'} \tag{37}$$

estimate

$$\delta \cdot \left\|\frac{d\nu}{d\sigma}\right\|_{L^{16/9}(\tau)} \le \delta^{-1/8} \left(\int_\tau \left|\hat{F}_r(\xi)\right|^2 \sigma(d\xi)\right)^{\frac{1-\theta}{2}} \left(\int_S \left|\hat{F}_r(\xi)\right|^{\bar{p}'} \sigma(d\xi)\right)^{\theta/\bar{p}'} \tag{38}$$

and from (36) and the inequality dual to (31)

$$(38) \le \delta^{-1/8} \left(r^{\frac{1-\rho}{2}} \delta^{\frac{3-\rho}{2}}\right)^{1-\theta} \|F_r\|_{\bar{p}'}^{\theta} \le \delta^{-1/8} \delta^{\frac{3-\rho}{2}(1-\theta)} r^{\frac{1-\rho}{2}(1-\theta)} r^{-\rho\theta/\bar{p}}. \tag{39}$$

Since $\rho \le 2$, the exponent of δ in (39) is positive and we let thus $\delta = 1$. Hence

$$\left\|\hat{F}_r|_S\right\|_{L^2(S)} \ll r^{-\frac{\rho}{6} + \frac{1-\rho}{6}(1-\theta) - \frac{\rho\theta}{3\bar{p}} + \varepsilon}. \tag{40}$$

Condition (24) becomes

$$\frac{\rho}{6} - \frac{1-\rho}{6}(1-\theta) + \frac{\rho\theta}{3\bar{p}} > \frac{3-\rho}{2}. \tag{41}$$

Letting $\bar{p} = \frac{58}{15}$, (37) determines θ and from (41) the condition on ρ becomes

$$\rho > \frac{1091}{546} = 1.998\ldots \tag{42}$$

(f) *Final remark.* Defining $\delta(\mu)$ as the image measure under the distance map $|x-y|$ of $\mu \times \mu$ (following [M]), the main point in the preceding is to obtain (for $n = 2, 3$) the property

$$\delta(\mu) \in L^2 \tag{43}$$

assuming $I_\alpha(\mu) = \iint |x-y|^{-\alpha} \mu(dx)\mu(dy) < \infty$ for some $\alpha < \frac{n+1}{2}$. This is the problem considered in remark 4.10 from [M].

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